

ON STABILITY OF N -TIMES INTEGRATED SEMIGROUPS WITH NONQUASIANALYTIC GROWTH

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ABSTRACT. We discuss the behaviour at infinity of n -times integrated semigroups with nonquasianalytic growth and invertible generator. The results obtained extend in this setting a theorem of O. El Mennaoui on stability of bounded once integrated semigroups, and (partially) a theorem of Q. P. Vũ on stability of C_0 -semigroups.

1. INTRODUCTION

Let A be a closed operator on a Banach space X with domain $D(A)$. The solution $u: [0, \infty) \rightarrow X$ of the Cauchy equation

$$(1) \quad u'(t) = Au(t), \quad t \geq 0; \quad u(0) = x \in D(A)$$

is given by $u(t) = T_0(t)x$ where $T_0(t) := e^{tA}$, $t \geq 0$, is the C_0 -semigroup generated by A , provided A satisfies the Hille-Yosida condition; see [2, Section 3.1]. There still are other important cases where A does not satisfy that condition but it is the generator of an exponentially bounded n -times integrated semigroup in the following sense:

There exist a family $(T_n(t))_{t \geq 0}$ of bounded operators on X and $C, w \geq 0$ such that $\|T_n(t)\| \leq Ce^{wt}$, $t \geq 0$, and

$$(\lambda - A)^{-1}x = \lambda^n \int_0^\infty e^{-t\lambda} T_n(t)x \, dt, \quad \Re \lambda > w, \quad x \in X.$$

Every C_0 -semigroup is a 0-times integrated semigroup; for more information on integrated semigroups and examples see [2, Sections 3.2 and 8.3], [3] and references therein.

Put $u(t) := (d/dt)^n T_n(t)x$, so that

$$T_n(t)x = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} u(s)x \, ds, \quad x \in X.$$

Then the function $u(t)$ is the unique solution to the equation (1) and the limit $\lim_{t \rightarrow \infty} T_n(t)$ -or alternatively its ergodic version $\lim_{t \rightarrow \infty} t^{-n} T_n(t)$ -

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reflects the asymptotic behaviour of the solution u at infinity. In this respect, when $n = 0$, a uniformly bounded C_0 -semigroup $(T_0(t))_{t \geq 0}$ is said to be stable if

$$\lim_{t \rightarrow \infty} T_0(t)x = 0, \quad x \in X.$$

The stability of uniformly bounded C_0 -semigroups on Banach spaces, under certain spectral assumptions on their infinitesimal generators, is proven in [1] and [8], with different proofs and independently one from each other. We refer to this stability result as the Arendt-Batty-Lyubich-Vũ theorem. It states that $(T_0(t))_{t \geq 0}$ is stable whenever

$$\sigma(A) \cap i\mathbb{R} \text{ is countable and } \sigma_P(A^*) \cap i\mathbb{R} = \emptyset.$$

Here $\sigma(A)$ is the spectrum of the generator A and $\sigma_P(A^*)$ is the point spectrum of the adjoint operator A^* of A . For the asymptotic behaviour and stability of operator semigroups we refer the reader to [4] and [9].

It seems interesting to have a result like the Arendt-Batty-Lyubich-Vũ theorem for n -times integrated semigroups. In this setting, a notion of stability has been defined for once integrated semigroups as follows. Suppose that $(T_0(t))_{t \geq 0}$ is a C_0 -semigroup and let $(T_1(t))_{t \geq 0}$ denote the (trivial, say) once integrated semigroup induced by $(T_0(t))_{t \geq 0}$, which is defined by

$$T_1(t)x := \int_0^t T_0(s)x \, ds, \quad x \in X.$$

By a well known property of C_0 -semigroups,

$$T_0(t)x - x = AT_1(t)x (= T_1(t)Ax), \quad \text{for } x \in D(A).$$

Assume in addition that the C_0 -semigroup $(T_0(t))_{t \geq 0}$ is stable. Then, *provided A is invertible*, one gets that there exists the limit

$$\lim_{t \rightarrow \infty} T_1(t)x = -A^{-1}x, \quad x \in \overline{D(A)}.$$

Motivated by this observation, a (nontrivial) general once integrated semigroup $T_1(t)$ is called *stable* in [5, p. 363] when $\lim_{t \rightarrow \infty} T_1(t)x$ exists for every $x \in \overline{D(A)}$. Moreover, it is also shown in [5, Prop. 5.1] that if a once integrated semigroup $T_1(t)$ is stable in the sense of that definition then A must be invertible, which is to say that 0 belongs to the resolvent set $\rho(A)$ of A . The following result is [5, Theorem 5.6]. It gives a version of the Arendt-Batty-Lyubich-Vũ theorem for once integrated semigroups.

Theorem 1.1. *Let A be the generator of a once integrated semigroup $(T_1(t))_{t \geq 0}$ such that $\sup_{t > 0} \|T_1(t)\| < +\infty$. Assume in addition that $\sigma(A) \cap i\mathbb{R}$ is countable, $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$ and $0 \in \rho(A)$. Then $(T_1(t))_{t \geq 0}$ is stable.*

The boundedness condition assumed on $(T_1(t))_{t \geq 0}$ in Theorem 1.1 looks somehow restrictive: For a uniformly bounded C_0 -semigroup $(T_0(t))_{t \geq 0}$ and its n -times integrated semigroup

$$T_n(t)x = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} T_0(s)x \, ds, \quad t > 0, x \in X,$$

the derived boundedness condition on $(T_n(t))_{t \geq 0}$ which is to be expected from the integral expression is $\sup_{t > 0} t^{-n} \|T_n(t)\| < +\infty$, so that for $n = 1$ is $\sup_{t > 0} t^{-1} \|T_1(t)\| < +\infty$ instead of boundedness.

The purpose of this note is to extend Theorem 1.1 to n -times integrated semigroups for every natural n and a fairly wide boundedness condition involving nonquasianalytic weights. We say that a positive measurable locally bounded function ω with domain \mathbb{R} or $[0, \infty)$ is a weight if $\omega(t) \geq 1$ and $\omega(s+t) \leq \omega(s)\omega(t)$ for all t, s in its domain. A weight ω on $[0, \infty)$ is called nonquasianalytic if

$$\int_0^\infty \frac{\log \omega(t)}{t^2 + 1} \, dt < \infty.$$

As in [11, Section 1] we assume that $\liminf_{t \rightarrow \infty} \omega(t)^{-1} \omega(s+t) \geq 1$ for all $s > 0$. Then one can define the function $\tilde{\omega}$ on \mathbb{R} given by

$$\tilde{\omega}(s) := \limsup_{t \rightarrow \infty} \frac{\omega(t+s)}{\omega(t)}, s \geq 0, \text{ and } \tilde{\omega}(s) := 1, s < 0,$$

is a weight function. Clearly, $\tilde{\omega}(t) \leq \omega(t)$ for every $t \geq 0$.

Our main result is the following. In the statement, and throughout the paper, the symbol “ \sim ” in $a(t) \sim b(t)$ as $t \rightarrow \infty$ means that $\lim_{t \rightarrow \infty} b(t)^{-1}a(t) = c > 0$ as $t \rightarrow \infty$.

Theorem 1.2. *Let A be the generator of a n -times integrated semigroup $(T_n(t))_{t \geq 0}$ such that $\sigma(A) \cap i\mathbb{R}$ is countable, $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$ and $0 \in \rho(A)$. Assume that*

$$\sup_{t \geq 1} \omega(t)^{-1} \|T_n(t)\| < +\infty,$$

for some nonquasianalytic weight ω on $[0, \infty)$ for which $\tilde{\omega}(t) = O(t^k)$, as $t \rightarrow \infty$, for some $k \geq 0$.

We have:

(i) *If $\omega(t)^{-1} = o(t^{-n+1})$ as $t \rightarrow \infty$, then*

$$\lim_{t \rightarrow \infty} \omega(t)^{-1} T_n(t)x = 0, \quad x \in \overline{D(A^n)}.$$

(ii) *If $\omega(t) \sim t^{n-1}$ as $t \rightarrow \infty$, then*

$$\lim_{t \rightarrow \infty} t^{-n+1} T_n(t)x = -\frac{1}{(n-1)!} A^{-1}x, \quad x \in \overline{D(A^n)}.$$

Remark 1.1. For $n = 1$, Theorem 1.2 (ii) is [5, Theorem 5.6]. So any n -times integrated semigroup $(T_n(t))_{t \geq 0}$ satisfying the equality of Theorem 1.2 (ii) might well be called *stable*. Alternatively, the ergodic

type equality $\lim_{t \rightarrow \infty} \omega(t)^{-1} T_n(t)x = 0$, $x \in \overline{D(A^n)}$, for $\omega(t) \sim t^n$ at infinity, defines a property on $(T_n(t))_{t \geq 0}$ which corresponds to stability of C_0 -semigroups when $n = 0$. Then one could say that a n -times integrated semigroup satisfying Theorem 1.2 (i) for $\omega(t) \sim t^n$ as $t \rightarrow \infty$ is *stable of order n* , and *stable under ω* in general.

2. PROOF OF THEOREM 1.2

In order to establish Theorem 1.2 one needs to extend [11, Theorem 7] and [5, Theorem 5.6]. Firstly, and more precisely, Theorem 2.1 below is an improvement of [11, Theorem 7], which is in turn an extension of the Arendt-Batty-Lyubich-Vũ theorem. In fact [11, Theorem 7] is recovered in the particular case that the operator R is the identity operator in Theorem 2.1. The case $\beta(t) \equiv 1$ in Theorem 2.1 appears in [1, Remark 3.3].

For a Banach space $(Y, \|\cdot\|)$, let $\mathcal{B}(Y)$ denote the Banach algebra of bounded operators on Y .

Theorem 2.1. *Let $(U(t))_{t \geq 0} \subset \mathcal{B}(Y)$ be a C_0 -semigroup of positive exponential type with generator L . Let β be a nonquasianalytic weight on $[0, \infty)$ such that $\tilde{\beta}(t) = O(t^k)$ as $t \rightarrow \infty$, for some $k \geq 0$. Assume that there exists $R \in \mathcal{B}(Y)$ such that $U(t)R = RU(t)$ for all $t \geq 0$ and $\|U(t)R\| \leq \beta(t)$ for $t \geq 0$.*

If $\sigma(L) \cap i\mathbb{R}$ is countable and $\sigma_P(L^) \cap i\mathbb{R} = \emptyset$ then*

$$\lim_{t \rightarrow \infty} \frac{1}{\beta(t)} U(t)Ry = 0, \quad y \in Y.$$

Proof. The overall argument goes along similar lines as in [11, Theorem 7], lemmata included. Next, we outline that argument for convenience of prospective readers and give details, when necessary, to extend the corresponding assertions to our setting.

Put

$$q(y) := \limsup_{t \rightarrow \infty} \beta(t)^{-1} \|U(t)Ry\|, \quad y \in Y.$$

Then q is a seminorm on Y such that $q(y) \leq \|y\|$ for all $y \in Y$. Moreover, $q(U(s)y) \leq \tilde{\beta}(s)q(y)$ for every $s \geq 0$, $y \in Y$, and so $N := \{y \in Y : q(y) = 0\}$ is a $U(t)$ -invariant closed subspace of Y . Hence one can define a norm \hat{q} on Y/N given by

$$\hat{q}(\pi(y)) := q(y), \quad y \in Y,$$

and an operator $\hat{U}(t)$ on Y/N given by

$$\hat{U}(t)(\pi(y)) := \pi(U(t)y), \quad y \in Y, t \geq 0,$$

where π is the projection $Y \rightarrow Y/N$.

It is straightforward to show that $(\hat{U}(t))_{t \geq 0}$ is a strongly continuous semigroup in the norm \hat{q} on Y/N . Let $(Z, \|\cdot\|_Z)$ be the \hat{q} -completion

of Y/N , and let $V(t)$ be the continuous extension on Z of $\widehat{U}(t)$ for all $t > 0$. Then:

- (a) $\|\pi(y)\|_Z = \limsup_{t \rightarrow \infty} \frac{1}{\beta(t)} \|U(t)Ry\|$ for $y \in Y$. This is obvious.
- (b) $\|V(t)\|_{Z \rightarrow Z} \leq \widetilde{\beta}(t)$, $t \geq 0$, and from this fact one readily obtains that $(V(t))_{t \geq 0}$ is a C_0 -semigroup in $\mathcal{B}(Z)$. The above bound follows by continuity and density from the estimate

$$\begin{aligned} \widehat{q}(\widehat{U}(t)\pi(y)) &= \widehat{q}(\pi(U(t)y)) = q(U(t)y) \\ &\leq \widetilde{\beta}(t)q(y) \leq \widetilde{\beta}(t)\widehat{q}(\pi(y)), \quad y \in Y, t \geq 0. \end{aligned}$$

- (c) $\|V(t)z\|_Z \geq \|z\|_Z$ for all $z \in Z$: For $y \in Y$ and $t \geq 0$,

$$\widehat{q}(\widehat{U}(t)\pi(y)) = \limsup_{t \rightarrow \infty} \frac{\beta(t+s)}{\beta(t)} \frac{\|U(t+s)Ry\|_Y}{\beta(t+s)} \geq \widehat{q}(\pi(y)).$$

Then we apply continuity and density.

- (d) $V(t) \circ \pi = \pi \circ U(t)$ ($t \geq 0$) and then one easily obtains that $\pi(D(L)) \subseteq D(H)$ and $H \circ \pi = \pi \circ L$ on $D(L)$, where H is the infinitesimal generator of $(V(t))_{t \geq 0}$.
- (e) $\sigma(H) \subseteq \sigma(L)$: By hypothesis, $(U(t))_{t \geq 0}$ is of exponential type $\delta > 0$ whence, as is well known, for $y \in Y$ and $\lambda \in \mathbb{C}$, $\Re \lambda > \delta$,

$$R(\lambda, L)y := -(\lambda - L)^{-1}y = - \int_0^\infty e^{-\lambda t} U(t)y \, dt.$$

Similarly, since $\|V(t)\|_{Z \rightarrow Z} \leq \widetilde{\beta}(t)$ for all $t \geq 0$, the semigroup $(V(t))_{t \geq 0}$ is of exponential type 0, and therefore we have for $z \in Z$ and $\lambda \in \mathbb{C}$, $\Re \lambda > 0$,

$$R(\lambda, H)z := -(\lambda - H)^{-1}z = - \int_0^\infty e^{-\lambda t} V(t)z \, dt.$$

On the other hand, R commutes with $U(t)$, $t \geq 0$, by assumption and so R commutes with $R(\lambda, L)$ for $\Re \lambda > \delta$. Then $q(R(\lambda, L)y) \leq \|R(\lambda, L)\|q(y)$ for all $y \in Y$, which implies that N is $R(\lambda, L)$ -invariant. Hence one can define the bounded operator $\widehat{R}(\lambda, L)$ on Z given by $\widehat{R}(\lambda, L)(\pi(y)) := \pi(R(\lambda, L)y)$, $y \in Y$. Thus,

$$\begin{aligned} \widehat{R}(\lambda, L)\pi(y) &= \pi(R(\lambda, L)y) = - \int_0^\infty e^{-\lambda t} \pi(U(t)y) \, dt \\ &= - \int_0^\infty e^{-\lambda t} V(t)\pi(y) \, dt = R(\lambda, H)\pi(y) \end{aligned}$$

where (d) has been applied in the last but one equality. Hence $\widehat{R}(\lambda, L) = R(\lambda, H)$, for $\Re \lambda > \delta$.

Now, for $\Re \lambda > \delta$ and any $\mu \in \rho(L)$, by using the resolvent identity

$$R(\lambda, L) - R(\mu, L) = (\lambda - \mu)R(\lambda, L)R(\mu, L)$$

on Y and its corresponding identity for $\widehat{R}(\lambda, L)$ and $\widehat{R}(\mu, L)$ on Z , one readily finds that there exists $R(\mu, H)$ with

$$R(\mu, H) = \widehat{R}(\mu, L),$$

see more details in [11, p. 234]. Thus $\mu \in \rho(H)$. Hence $\rho(L) \subseteq \rho(H)$ as we claimed.

(f) $\sigma_P(H^*) \subseteq \sigma_P(L^*)$. This is straightforward to see, using restrictions of functionals; see [11, p. 234].

Suppose, if possible, that $Z \neq \{0\}$. By (e) above, we have that $\sigma(H) \cap i\mathbb{R}$ is countable and then $i\mathbb{R} \setminus \sigma(H) \neq \emptyset$. So, by (c) above and [11, Lemma 2], the C_0 -semigroup $(V(t))_{t \geq 0}$ can be extended to a C_0 -group $(\widetilde{V}(t))_{t \in \mathbb{R}}$ such that $\|\widetilde{V}(-t)\|_{Z \rightarrow Z} \leq 1$ ($t > 0$) and $\|\widetilde{V}(t)\|_{Z \rightarrow Z} = O(t^k)$, as $t \rightarrow +\infty$. Also, $\sigma(H)$ is nonempty by (b) above and [11, Lemma 5].

Then reasoning as in [11, Theorem 7] one gets $\sigma_P(H^*) \cap i\mathbb{R} \neq \emptyset$ whence $\sigma_P(L^*) \cap i\mathbb{R} \neq \emptyset$ by (f) above. This is a contradiction and so we have proved that $Z = \{0\}$. By (a) above, we get the statement. \square

The following theorem is the quoted extension of [5, Theorem 5.6].

Theorem 2.2. *Let ω be a nonquasianalytic weight such that $\widetilde{\omega}$ is of polynomial growth at infinity. Let $(X, \|\cdot\|)$ be a Banach space and $(T_n(t))_{t \geq 0}$ be a n -times integrated semigroup in $\mathcal{B}(X)$ with generator $(A, D(A))$ such that $\|T_n(t)\| \leq \omega(t)$, $t \geq 0$. Let assume that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$.*

For every $\mu > 0$ we have:

(i) *If $\omega(t)^{-1} = o(t^{-(n-1)})$, as $t \rightarrow \infty$, then*

$$\lim_{t \rightarrow \infty} \omega(t)^{-1} T_n(t) A^n (\mu - A)^{-2n} x = 0, \quad x \in X.$$

(ii) *If $\omega(t) \sim t^{n-1}$, as $t \rightarrow \infty$, then*

$$\lim_{t \rightarrow \infty} \frac{1}{t^{n-1}} T_n(t) A^n (\mu - A)^{-2n} x = -\frac{A^{n-1} (\mu - A)^{-n} x}{(n-1)!}, \quad x \in X.$$

Proof. Take $\mu > \delta > 0$. For $x \in X$ define

$$\|x\|_Y := \sup_{t \geq 0} \|e^{-\delta t} (T_n(t) A^n (\mu - A)^{-n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-n} x)\|.$$

Note that $A(\mu - A)^{-1} = -I + \mu(\mu - A)^{-1}$ is a bounded operator on X and $T_n(0) = 0$, so $\|\cdot\|_Y$ is a norm on X and there exists a constant $M_\delta > 0$ such that

$$(2) \quad \|(\mu - A)^{-n} x\| \leq \|x\|_Y \leq M_\delta \|x\|, \quad x \in X.$$

Let Y be the Banach space obtained as the completion of X in the norm $\|\cdot\|_Y$. By the Extrapolation Theorem [3, Theorem 0.2], there exists a closed operator B on Y which generates a C_0 -semigroup $(S(t))_{t \geq 0} \subset \mathcal{B}(Y)$ of positive exponential type such that $D(B^n) \hookrightarrow X \hookrightarrow Y$, $A = B_X$ where the operator B_X is given by the conditions

$D(B_X) := \{x \in D(B) \cap X : Bx \in X\}$, $B_X(x) := B(x)$ ($x \in X$). Moreover, $\sigma_P(B^*) \subseteq \sigma_P(A^*)$, and also $\rho(A) = \rho(B)$ with

$$(3) \quad (\lambda - A)^{-1}x = (\lambda - B)^{-1}x, \quad \lambda \in \rho(A) = \rho(B), x \in X;$$

see [3, Remark 3.1].

Let $(S_n(t))_{t \geq 0}$ be the n -times integrated semigroup generated by B on Y , given by

$$S_n(t)y := \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} S(s)y \, ds, \quad y \in Y.$$

Then $S_n(t)x = T_n(t)x$ for all $x \in X$ and $t \geq 0$. To see this, note that $(T_n(t))_{t \geq 0}$ and $(S_n(t))_{t \geq 0}$ are of exponential type so one can rewrite (3) above in terms of the Laplace transforms of $(T_n(t))_{t \geq 0}$ and $(S_n(t))_{t \geq 0}$ respectively, for $\Re \lambda$ large enough. Then it suffices to apply the uniqueness of the Laplace transform.

From the above identification between $T_n(t)$ and $S_n(t)$, it readily follows that

$$\|S_n(u)x\|_Y \leq \|T_n(u)\| \|x\|_Y \leq \omega(u)\|x\|_Y, \quad u \geq 0, x \in X,$$

which is to say, by density, that $\|S_n(u)\| \leq \omega(u)$, for all $u \geq 0$.

Now, by reiteration of the well known equality

$$S(t)y - y = \int_0^t BS(s)y \, ds \quad (t \geq 0, y \in D(B)),$$

we have

$$S(t)y = S_n(t)B^n y + \sum_{j=0}^{n-1} \frac{t^j}{j!} B^j y, \quad y \in D(B^n).$$

Hence, for every $y \in Y$,

$$(4) \quad S(t)(\mu - B)^{-n}y = S_n(t) \left(\frac{B}{\mu - B} \right)^n y + \sum_{j=0}^{n-1} \frac{t^j}{j!} \left(\frac{B}{\mu - B} \right)^j (\mu - B)^{-(n-j)}y$$

and therefore there exists a constant $C_\mu > 0$ such that

$$\|S(t)(\mu - B)^{-n}\|_{Y \rightarrow Y} \leq C_\mu \omega(t), \quad t \geq 0.$$

Then, by applying Theorem 1.2 with $U(t) = S(t)$, $B = L$ and $R = (\mu - A)^{-n}$, we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{\omega(t)} \|S(t)(\mu - B)^{-n}y\|_Y = 0, \quad y \in Y,$$

whence, by (2), (3) and (4),

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \frac{1}{\omega(t)} \|T_n(t) A^n (\mu - A)^{-n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-n} x\|_Y \\ &\geq \limsup_{t \rightarrow \infty} \frac{1}{\omega(t)} \|T_n(t) A^n (\mu - A)^{-2n} x + \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-2n} x\|_X, \end{aligned}$$

for every $x \in X$.

Thus we get

$$\lim_{t \rightarrow \infty} \frac{1}{\omega(t)} T_n(t) A^n (\mu - A)^{-2n} x = - \lim_{t \rightarrow \infty} \frac{1}{\omega(t)} \sum_{j=0}^{n-1} \frac{t^j}{j!} A^j (\mu - A)^{-2n} x$$

in X , and the statement follows readily. \square

Proof of Theorem 1.2. In the setting of Theorem 2.2, let assume in addition that $0 \in \rho(A)$. Since the resolvent function of A is holomorphic -so continuous- in the open subset $\rho(A) \subseteq \mathbb{C}$ we have that

$$\lim_{\mu \rightarrow 0^+} A^n (\mu - A)^{-n} = \lim_{\mu \rightarrow 0^+} (-I + \mu(\mu - A)^{-1})^n = (-1)^n I.$$

Now, to prove (i) and (ii) of the theorem it suffices to notice that $\sup_{t > 0} \omega(t)^{-1} \|T_n(t)\| < \infty$ in both cases. \square

3. FINAL COMMENTS AND REMARKS

It looks desirable to find out the behavior of a n -times integrated semigroup at infinity when its generator A is not assumed to be invertible. According to the remark prior to Theorem 1.1 the existence of $\lim_{t \rightarrow \infty} T_n(x)$ (for $n = 1$) entails invertibility of A . Thus the type of convergence at infinity of $T_n(t)$, if there is some, that one can expect if A is not invertible must be weaker than the existence of limit.

In [7], under the assumptions

$$\sup_{t > 0} t^{-n} \|T_n(t)\| < \infty \text{ and } \lim_{t \rightarrow 0^+} n! t^{-n} T_n(t) x = x \quad (x \in X),$$

it has been proved that

$$\lim_{t \rightarrow \infty} t^{-n} T_n(t) \pi_n(f) = 0, \quad f \in \mathfrak{S}_n,$$

in the operator norm, where \mathfrak{S}_n is the subspace of functions of $\mathcal{T}_+^{(n)}(t^n)$ which are of spectral synthesis in $\mathcal{T}^{(n)}(|t|^n)$ with respect to the subset $i\sigma(A) \cap \mathbb{R}$, and $\pi_n(f) = (-1)^n \int_0^\infty f^{(n)}(t) T_n(t) dt$. Here, $\mathcal{T}^{(n)}(|t|^n)$ is the convolution Banach algebra obtained as the completion of the Schwarz class in the norm $f \mapsto \int_{-\infty}^\infty |f^{(n)}(t)| |t|^n dt$, and $\mathcal{T}_+^{(n)}(t^n)$ is the restriction of $\mathcal{T}^{(n)}(|t|^n)$ on $(0, \infty)$. This result is an extension of the Esterle-Strouse-Vũ-Zouakia theorem, which corresponds to the case $n = 0$; see [6] and [10]. In [6] it is shown that, under the assumptions that $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$, the subspace $\pi_0(\mathfrak{S}_0)X$

is dense in X so one gets another way –for it is different from the original one– to establish the Arendt-Batty-Lyubich-Vũ theorem. The proof of that density is attained by methods of harmonic analysis.

We wonder if in the case when A is not invertible the argument considered in [6] to deduce the Arendt-Batty-Lyubich-Vũ theorem works for n -times integrated semigroups; that is, if $\pi_n(\mathfrak{S}_n)X$ is dense in X (under the conditions $\sigma(A) \cap i\mathbb{R}$ countable and $\sigma_P(A^*) \cap i\mathbb{R} = \emptyset$). This would give us the ergodic type property

$$(5) \quad \lim_{t \rightarrow \infty} t^{-n} T_n(t)x = 0, \quad x \in X.$$

Notice that (5) is a consequence of Theorem 1.2 (i) when A is invertible; on the other hand, the ergodicity of a n -times integrated semigroup $(T_n(t))_{t \geq 0}$ such that $\sup_{t \geq 1} t^{-n} \|T_n(t)\| < \infty$ is characterized in [5] in terms of Abel-ergodicity or/and ergodic decompositions of the Banach space X . Such an approach will be considered in a forthcoming paper.

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